

YITP-00-40
hep-th/0007200
July 2000

Comment on Ambiguities in the Holographic Weyl Anomaly

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ABSTRACT

We consider possible ambiguities in the holographic Weyl anomaly that may arise from local terms in the flow equation. We point out that such ambiguities actually do not give physically meaningful contributions to the Weyl anomaly.

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The AdS/CFT correspondence [1] (for a review see Ref. [2]) provides us with a useful framework to study the renormalization group (RG) structure of boundary field theories [3][4][5][6][7][8][9][10][11]. In a remarkable paper [12], de Boer, Verlinde and Verlinde reformulated the holographic RG on the basis of the Hamilton-Jacobi equation for gravity systems, and introduced the “flow equation,” from which one can easily derive the Weyl anomaly of boundary conformal field theories, that is in exact agreement with the one obtained in Ref. [13]. The property of the flow equation is explored in detail in Ref. [14], and a prescription for calculating the holographic Weyl anomaly is given for arbitrary dimensions. In this short note, we study possible ambiguities in the prescription that may arise from local terms in the flow equation, and point out that they actually do not yield physically meaningful effects to the Weyl anomaly. This result is not new and may have been noticed for experts. However, we believe that it is instructive to discuss it in some detail because this point can be clarified in a simple manner by the formulation given in Ref. [14]. We discuss bulk gravity with scalar fields. The extension to other cases should be straightforward.

We start with the bulk action for a $(d+1)$ -dimensional manifold M_{d+1} :

$$S_{d+1}[\mathbf{G}_{MN}(x, r), \phi^i(x, r)] = \int_{M_{d+1}} d^{d+1}X \sqrt{\mathbf{G}} \left(V(\phi) - \mathbf{R} + \frac{1}{2} L_{ij}(\phi) \mathbf{G}^{MN} \partial_M \phi^i \partial_N \phi^j \right) - 2 \int_{\Sigma_d} d^d x \sqrt{G} K, \quad (1)$$

where $X^M = (x^\mu, r)$ with $\mu, \nu = 1, 2, \dots, d$. The Euclidean time r is regarded as the flow parameter of the RG, and we have introduced a UV cut-off r_0 such that $r_0 \leq r < \infty$. The second term in eq. (1) is the contribution from the boundary $\Sigma_d \equiv \partial M_{d+1}$ at $r = r_0$, which needs to be introduced in order for Dirichlet boundary conditions to be imposed consistently [15]. In what follows, we take the bulk metric \mathbf{G}_{MN} to be in the temporal gauge [12]

$$ds^2 = \mathbf{G}_{MN} dX^M dX^N = dr^2 + G_{\mu\nu}(x, r) dx^\mu dx^\nu, \quad (2)$$

for which the extrinsic curvature is given as $K = (1/2) G^{\mu\nu} \partial_r G_{\mu\nu}$. Let $\bar{G}_{\mu\nu}(x, r; G(x), r_0)$ and $\bar{\phi}^i(x, r; \phi(x), r_0)$ be the classical trajectory of $G_{\mu\nu}(x, r)$ and $\phi^i(x, r)$ with the boundary condition

$$\bar{G}_{\mu\nu}(x, r=r_0) = G_{\mu\nu}(x), \quad \bar{\phi}^i(x, r=r_0) = \phi^i(x). \quad (3)$$

We assume that the classical trajectory is uniquely determined by this initial value, demanding the regular behavior inside M_{d+1} ($r \rightarrow +\infty$). The on-shell action $S[G(x), \phi(x)]$ is then defined as a functional of the boundary values and obtained by substituting the classical solution into the bulk action

$$S[G_{\mu\nu}(x), \phi(x)] \equiv S_{d+1} \left[\bar{G}_{\mu\nu}(x, r; G(x), r_0), \bar{\phi}^i(x, r; \phi(x), r_0) \right]. \quad (4)$$

Reflecting general covariance along the r -direction, the on-shell action does not depend on the coordinate value of the boundary, r_0 , and obeys the *flow equation* [12]

$$\{S, S\}(x) = \sqrt{G(x)} \mathcal{L}_d(x), \quad (5)$$

where

$$\mathcal{L}_d(x) \equiv V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j, \quad (6)$$

and for arbitrary functionals A and B , $\{A, B\}(x)$ is defined by

$$\{A, B\}(x) \equiv \frac{1}{\sqrt{G}} \left[\left(-\frac{1}{d-1} G_{\mu\nu} G_{\rho\sigma} + G_{\mu\rho} G_{\nu\sigma} \right) \frac{\delta A}{\delta G_{\mu\nu}} \frac{\delta B}{\delta G_{\rho\sigma}} + \frac{1}{2} L^{ij}(\phi) \frac{\delta A}{\delta \phi^i} \frac{\delta B}{\delta \phi^j} \right]. \quad (7)$$

To solve the flow equation (5)–(7), we first decompose the on-shell action as

$$S[G(x), \phi(x)] = S_{\text{loc}}[G(x), \phi(x)] + \Gamma[G(x), \phi(x)], \quad (8)$$

where S_{loc} is the local part that can be written as an integral of a local function, and Γ gives the generating functional of the boundary field theory [12]. Note that there is an ambiguity in this decomposition, reflecting that one can add any local terms to Γ . However, by introducing the following weight w and by assigning the vanishing weight to Γ [14], one can make this decomposition unique up to additions of local terms of weight 0:

	weight w
$G_{\mu\nu}(x), \phi^i(x), \Gamma[G, \phi]$	0
∂_μ	1
$R, R_{\mu\nu}, \partial_\mu \phi^i \partial_\nu \phi^j, \dots$	2
$\delta \Gamma / \delta G_{\mu\nu}(x), \delta \Gamma / \delta \phi^i(x)$	d

S_{loc} is then expanded with respect to this weight:

$$S_{\text{loc}}[G(x), \phi(x)] = \int d^d x \sqrt{G(x)} \sum_{w=0,2,4,\dots} [\mathcal{L}_{\text{loc}}(x)]_w. \quad (9)$$

The first few terms can be parametrized as

$$[\mathcal{L}_{\text{loc}}]_0 = W(\phi), \quad [\mathcal{L}_{\text{loc}}]_2 = -\Phi(\phi) R + \frac{1}{2} M_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j. \quad (10)$$

One can show that $[\mathcal{L}_{\text{loc}}]_0, \dots, [\mathcal{L}_{\text{loc}}]_{d-2}$ are determined by the flow equation at weight $w = 0, \dots, d-2$, and can be written in terms of $V(\phi)$ and $L_{ij}(\phi)$ [12][14]. On the other hand, the flow equation at weight $w = d$ gives the following equation that would determine Γ :

$$-2 G_{\mu\nu} \frac{\delta \Gamma}{\delta G_{\mu\nu}} + \beta^i \frac{\delta \Gamma}{\delta \phi^i} = -\frac{1}{[\gamma]_0} [\{S_{\text{loc}}, S_{\text{loc}}\}]_d, \quad (11)$$

where

$$[\gamma]_0 = \frac{W(\phi)}{2(d-1)}, \quad \beta^i = \frac{2(d-1)}{W(\phi)} L^{ij}(\phi) \partial_j W(\phi). \quad (12)$$

The right-hand side of eq. (11) generally consists of the d -dimensional Weyl anomaly \mathcal{W}_d of the boundary field theory and the contribution from the $[\mathcal{L}_{\text{loc}}]_d$ [14]:

$$-\frac{1}{[\gamma]_0} [\{S_{\text{loc}}, S_{\text{loc}}\}]_d = -2 \sqrt{G} \mathcal{W}_d - \frac{2}{[\gamma]_0} \{S_{\text{loc};-d}, S_{\text{loc};0}\}, \quad (13)$$

where we have introduced

$$S_{\text{loc};w-d} \equiv \int d^d x \sqrt{G(x)} [\mathcal{L}_{\text{loc}}]_w. \quad (14)$$

The weight shifts by $-d$ after the integration because the weight of $d^d x$ is $-d$. Since the Weyl anomaly \mathcal{W}_d can be totally written in terms of $[\mathcal{L}_{\text{loc}}]_0, \dots, [\mathcal{L}_{\text{loc}}]_{d-2}$, eq. (11) shows that Γ can only be determined up to contributions from $[\mathcal{L}_{\text{loc}}]_d$. However, by using the relations

$$\frac{\delta S_{\text{loc};-d}}{\delta G_{\mu\nu}} = \frac{\sqrt{G}}{2} W(\phi) G^{\mu\nu}, \quad \frac{\delta S_{\text{loc};-d}}{\delta \phi^i} = \sqrt{G} \partial_i W(\phi), \quad (15)$$

one finds that

$$-\frac{1}{[\gamma]_0} [\{S_{\text{loc}}, S_{\text{loc}}\}]_d = -2 \sqrt{G} \mathcal{W}_d + 2 G_{\mu\nu} \frac{\delta S_{\text{loc};0}}{\delta G_{\mu\nu}} - \beta^i \frac{\delta S_{\text{loc};0}}{\delta \phi^i}, \quad (16)$$

and can rewrite the flow equation (11) as

$$-2 G_{\mu\nu} \frac{\delta}{\delta G_{\mu\nu}} (\Gamma + S_{\text{loc};0}) + \beta^i \frac{\delta}{\delta \phi^i} (\Gamma + S_{\text{loc};0}) = -2 \sqrt{G} \mathcal{W}_d. \quad (17)$$

Thus, we have seen that the contribution from the term $[\mathcal{L}_{\text{loc}}]_d$ can be absorbed into Γ by redefining it as $\Gamma' = \Gamma + S_{\text{loc};0}$. Note that Γ' still has vanishing weight.

Instead of redefining Γ , one can modify the Weyl anomaly without making any essential change. To show this, we first notice that the second term in eq. (16) can be written as a total derivative:

$$2 G_{\mu\nu} \frac{\delta S_{\text{loc};0}}{\delta G_{\mu\nu}} = -2 \sqrt{G} \nabla_\mu \mathcal{J}_d^\mu \quad (18)$$

with \mathcal{J}_d^μ some local current. In fact, for infinitesimal Weyl transformations ($\sigma(x) \ll 1$: arbitrary function), we have

$$S_{\text{loc};0}[e^{\sigma(x)} G(x), \phi(x)] - S_{\text{loc};0}[G(x), \phi(x)] = \int d^d x \sigma(x) G_{\mu\nu} \frac{\delta S_{\text{loc};0}}{\delta G_{\mu\nu}}. \quad (19)$$

One can easily understand that $S_{\text{loc};0}[G(x), \phi(x)]$ is invariant under *constant* Weyl transformations ($G_{\mu\nu}(x) \rightarrow e^\sigma G_{\mu\nu}(x)$, $\phi^i(x) \rightarrow \phi^i(x)$ with σ constant), so that the left-hand side of eq. (19) can generally be written as

$$\int d^d x \partial_\mu \sigma(x) \sqrt{G} \mathcal{J}_d^\mu \quad (20)$$

with some local function \mathcal{J}_d^μ . By integrating this by parts and comparing the result with the right-hand side of eq. (19), one obtains eq. (18). Thus we have shown that eq. (11) can be rewritten into the following form:

$$-2 G_{\mu\nu} \frac{\delta \Gamma}{\delta G_{\mu\nu}} + \beta^i \frac{\delta \Gamma}{\delta \phi^i} = -2 \sqrt{G} (\mathcal{W}_d + \nabla_\mu \mathcal{J}_d^\mu) - \beta^i \frac{\delta S_{\text{loc};0}}{\delta \phi^i}. \quad (21)$$

This implies that when we take Γ as the generating functional, the Weyl anomaly differs from \mathcal{W}_d only by a total derivative at conformal fixed points ($\beta^i = 0$).

In summary, in the formulation of the holographic RG based on the flow equation, the ambiguity in defining the generating functional Γ totally corresponds (as expected) to the addition of a total derivative term to the Weyl anomaly when the beta functions vanish.

Acknowledgment

The authors would like to thank S. Matsuura for a useful discussion. The work of M.F. is supported in part by a Grand-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture, and the work of T.S. is supported in part by JSPS Research Fellowships for Young Scientists.

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